1 LISREL models and methods

The LISREL model, in its most general form, consists of two parts: the measurement model and the structural equation model.

The **measurement model** specifies how latent variables or hypothetical constructs depend upon or are indicated by the observed variables. It describes the measurement properties (reliabilities and validities) of the observed variables.

The **structural equation model** specifies the causal relationships among the latent variables, describes the causal effects, and assigns the explained and unexplained variance.

The LISREL method estimates the unknown coefficients of a set of linear structural equations. It is particularly designed to accommodate models that include latent variables, measurement errors in both dependent and independent variables, reciprocal causation, simultaneity, and interdependence.

As implemented in the LISREL 7 program, the method includes as special cases such procedures as confirmatory factor analysis, multiple regression analysis, path analysis, economic models for time-dependent data, recursive and non-recursive models for cross-sectional and longitudinal data, and covariance structure models.

1.1 The general LISREL model

The full LISREL model for single samples is defined, for deviations about the mean, by the following three equations:

The structural equation model:

$$\eta = \mathbf{B}\eta + \mathbf{\Gamma}\xi + \zeta \tag{1.1}$$

The measurement model for $\mathbf{y}$:

$$\mathbf{y} = \mathbf{\Lambda}_y \eta + \epsilon \tag{1.2}$$

The measurement model for $\mathbf{x}$:

$$\mathbf{x} = \mathbf{\Lambda}_x \xi + \delta \tag{1.3}$$

The terms in these models are defined as follows:

- $\mathbf{y}$ is a $p \times 1$ vector of observed response or outcome variables.
- $\mathbf{x}$ is a $q \times 1$ vector of predictors, covariates, or input variables.
- η is an $m \times 1$ random vector of latent dependent, or endogenous, variables.
- ξ is an $n \times 1$ random vector of latent independent, or exogenous, variables.
- ϵ is a $p \times 1$ vector of measurement errors in $\mathbf{y}$.
- δ is a $q \times 1$ vector of measurement errors in $\mathbf{x}$.
- $\mathbf{\Lambda}_y$ is a $p \times m$ matrix of coefficients of the regression of $\mathbf{y}$ on η.
- $\mathbf{\Lambda}_x$ is a $q \times n$ matrix of coefficients of the regression of $\mathbf{x}$ on ξ.
- $\mathbf{\Gamma}$ is an $m \times n$ matrix of coefficients of the ξ-variables in the structural relationship.
- $\mathbf{B}$ is an $m \times m$ matrix of coefficients of the η-variables in the structural relationship. $\mathbf{B}$ has zeros in the diagonal, and $\mathbf{I} - \mathbf{B}$ is required to be non-singular.
- ζ is an $m \times 1$ vector of equation errors (random disturbances) in the structural relationship between η and ξ.

Assumptions

The random components in the LISREL model are assumed to satisfy the following minimal assumptions:

ϵ is uncorrelated with η
δ is uncorrelated with ξ
ζ is uncorrelated with ξ

ζ, ϵ, and δ are mutually uncorrelated

Covariance matrices:

$\mathrm{Cov}(\boldsymbol{\xi}) = \boldsymbol{\Phi}\ (n \times n)$ $\quad$ $\mathrm{Cov}(\boldsymbol{\zeta}) = \boldsymbol{\Psi}\ (m \times m)$
$\mathrm{Cov}(\boldsymbol{\epsilon}) = \boldsymbol{\Theta}_\epsilon\ (p \times p)$ $\quad$ $\mathrm{Cov}(\boldsymbol{\delta}) = \boldsymbol{\Theta}_\delta\ (q \times q)$

The covariance matrix of the observations as implied by the LISREL model

The assumptions in the previous section imply the following form for the covariance matrix of the observed variables:

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Lambda}_y \mathbf{A}(\boldsymbol{\Gamma\Phi\Gamma}' + \boldsymbol{\Psi})\mathbf{A}'\boldsymbol{\Lambda}_y' + \boldsymbol{\Theta}_\epsilon & \boldsymbol{\Lambda}_y \mathbf{A}\boldsymbol{\Gamma\Phi\Lambda}_x' \\ \boldsymbol{\Lambda}_x \boldsymbol{\Phi\Gamma}'\mathbf{A}'\boldsymbol{\Lambda}_y' & \boldsymbol{\Lambda}_x \boldsymbol{\Phi\Lambda}_x' + \boldsymbol{\Theta}_\delta \end{pmatrix} \tag{1.4}$$

where $\mathbf{A} = (\mathbf{I} - \mathbf{B})^{-1}$.

Fixed, free, and constrained parameters

The general LISREL model is specialized by fixing and constraining the parameters that comprise the elements in $\boldsymbol{\Lambda}_y$, $\boldsymbol{\Lambda}_x$, $\mathbf{B}$, $\boldsymbol{\Gamma}$, $\boldsymbol{\Phi}$, $\boldsymbol{\Psi}$, $\boldsymbol{\Theta}_\epsilon$ and $\boldsymbol{\Theta}_\delta$. The elements are of three kinds:

Fixed parameters — assigned specified values
Constrained parameters — unknown but equal to one or more other unknown parameters
Free parameters — unknown and not constrained to be equal to other parameters

LISREL notation for numbers of variables

Variables	*Number*	*Notation*
y	p	NY
x	q	NX
η	m	NE
ξ	n	NK

1.2 Path diagrams and the LISREL equations

A path diagram represents the relationship among variables in the LISREL model. If drawn and labeled correctly and in sufficient detail, the diagram can specify exactly the fixed, constrained or free status of all parameters in the model.

The following are rules for drawing path diagrams:

- The observed x- and y-variables are enclosed in boxes.
- The latent variables ξ and η are enclosed in circles or ellipses.
- The error variables ϵ, δ, and ζ appear in the diagram but are not enclosed.
- A one-way arrow between two variables indicate a postulated direct influence of one variable on another. A two-way arrow between two variables indicates that these variables may be correlated without any assumed direct relationship. One-way arrows are drawn straight, while two-way arrows are generally curved.
- There is a fundamental distinction between independent variables (ξ-variables) and dependent variables (η-variables). Variation and covariation in the dependent variables is to be accounted for or explained by the independent variables. In the path diagram this corresponds to the statements,

  1 no one-way arrow can point to a ξ-variable;

  2 all one-way arrows pointing to an η-variable come from ξ- and η-variables.

- Coefficients are associated with each arrow as follows:

  One-way arrows ($\rightarrow$):

  An arrow from ξ_i to x_b is denoted $\lambda_{bi}^{(x)}$.

  An arrow from η_g to y_a is denoted $\lambda_{ag}^{(y)}$.

  An arrow from η_h to η_g is denoted β_{gh}.

  An arrow from ξ_i to η_g is denoted γ_{gi}.

  Two-way arrows ($\leftrightarrow$):

  A two-way arrow from ξ_j to ξ_i is denoted ϕ_{ij}.

  A two-way arrow from ζ_h to ζ_g is denoted ψ_{gh}.

  A two-way arrow from δ_b to δ_a is denoted $\theta_{ab}^{(\delta)}$.

A two-way arrow from ϵ_d to ϵ_c is denoted $\theta_{cd}^{(\epsilon)}$.

- Each coefficient has two subscripts: the first is the subscript of the variable to which the arrow is pointing; the second is the subscript of the variable from which the arrow is coming. In Figure 1.1, for example, γ_{23} corresponds to the arrow from ξ_3 to η_2. For two-way arrows, the two subscripts may be interchanged ($\phi_{21} = \phi_{12}$ in Figure 1.1). Arrows that have no explicit coefficient in the path diagram are assumed to have a unit coefficient (unless space limits prevents their appearance).
- All direct influences of one variable on another must be included in the path diagram. The non-existence of an arrow between two variables means that the two variables are assumed not directly related. (They may still be indirectly related, however, see Section 1.10.)

If the above conventions for path diagrams are followed exactly, it is possible to write the corresponding model equations by means of the following general rules:

1. For each variable which has a one-way arrow pointing to it there will be one equation in which this variable is a left-hand variable.
2. The right-hand side of each equation is the sum of a number of terms equal to the number of one-way arrows pointing to that variable and each term is the product of the coefficient associated with the arrow and the variable from which the arrow is coming.

Equations for the path diagram

With these rules, the equations for the path diagram in Figure 1.1 can be written as follows.

The diagram shows there are seven x-variables as indicators of three latent ξ-variables. Note that x_3 is a variable measuring both ξ_1 and ξ_2. There are two latent η-variables each with two y-indicators. The five latent variables are connected in a two-equation interdependent system. The model involves errors in equations (the ζ's) and errors in variables (the ϵ's and δ's).

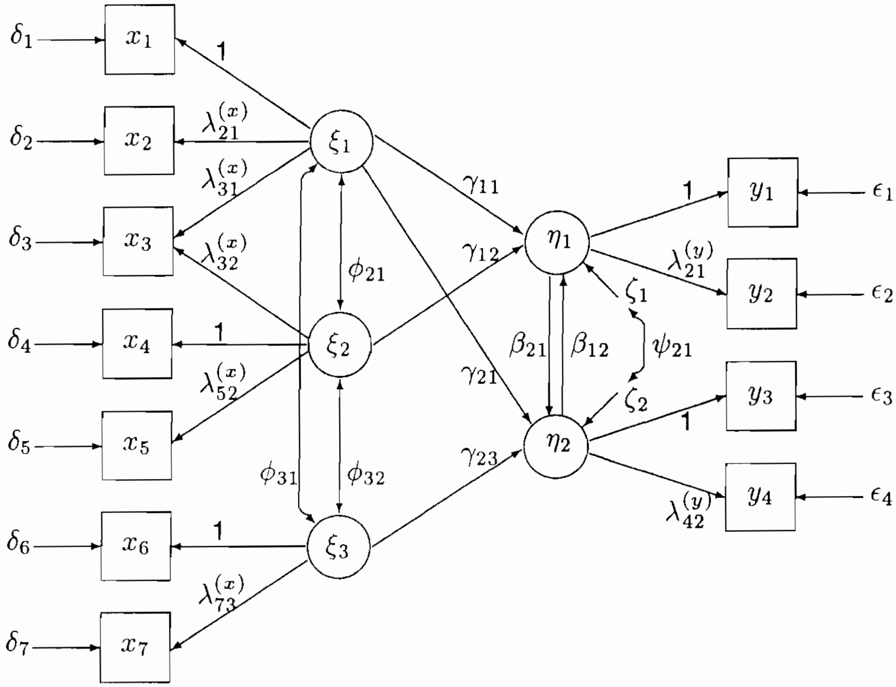

Figure 1.1 Path Diagram for Hypothetical Model

The *structural equations* are

$$\begin{aligned} \eta_1 &= \beta_{12}\eta_2 + \gamma_{11}\xi_1 + \gamma_{12}\xi_2 + \zeta_1 \\ \eta_2 &= \beta_{21}\eta_1 + \gamma_{21}\xi_1 + \gamma_{23}\xi_3 + \zeta_2 \end{aligned}$$

or in matrix form

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 & \beta_{12} \\ \beta_{21} & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} \gamma_{11} & \gamma_{12} & 0 \\ \gamma_{21} & 0 & \gamma_{23} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} + \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} .$$

Either of these forms corresponds to the structural equation model represented in Figure 1.1.

The *measurement model* equations for y-variables are

$$y_1 = \eta_1 + \epsilon_1$$

$$
\begin{aligned}
y_2 &= \lambda_{21}^{(y)} \eta_1 + \epsilon_2 \\
y_3 &= \eta_2 + \epsilon_3 \\
y_4 &= \lambda_{42}^{(y)} \eta_2 + \epsilon_4
\end{aligned}
$$

or in matrix form

$$
\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda_{21}^{(y)} & 0 \\ 0 & 1 \\ 0 & \lambda_{42}^{(y)} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{pmatrix} .
$$

The *measurement model* equations for x-variables are

$$
\begin{aligned}
x_1 &= \xi_1 + \delta_1 \\
x_2 &= \lambda_{21}^{(x)} \xi_1 + \delta_2 \\
x_3 &= \lambda_{31}^{(x)} \xi_1 + \lambda_{32}^{(x)} \xi_2 + \delta_3 \\
x_4 &= \xi_2 + \delta_4 \\
x_5 &= \lambda_{52}^{(x)} \xi_2 + \delta_5 \\
x_6 &= \xi_3 + \delta_6 \\
x_7 &= \lambda_{73}^{(x)} \xi_3 + \delta_7
\end{aligned}
$$

or in matrix form

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda_{21}^{(x)} & 0 & 0 \\ \lambda_{31}^{(x)} & \lambda_{32}^{(x)} & 0 \\ 0 & 1 & 0 \\ 0 & \lambda_{52}^{(x)} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \lambda_{73}^{(x)} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} + \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \end{pmatrix} .
$$

These equations correspond to the measurement models represented in Figure 1.1.

One λ in each column of $\mathbf{\Lambda}_y$ and $\mathbf{\Lambda}_x$ has been set equal to 1 to fix the scales of measurement in the latent variables.

In these equations, note that the second subscript on each coefficient is always equal to the subscript of the variable that follows the coefficient. This correspondence serves to check that the terms are correct.

In the matrices $\mathbf{B}$, $\mathbf{\Gamma}$, Λ_y, and Λ_x, the subscripts on each coefficient, as originally defined in the path diagram, correspond to the row and column of the matrix in which they appear. Note that possible paths that are *not* included in the diagram correspond to *zeros* in these matrices.

Each of the parameter matrices contain fixed elements (the zeros and ones) and free parameters (the coefficients with two subscripts).

The four remaining parameter matrices are symmetric matrices:

the covariance matrix of ξ,

$$\mathbf{\Phi} = \begin{pmatrix} \phi_{11} & & \\ \phi_{21} & \phi_{22} & \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix},$$

the covariance matrix of ζ,

$$\mathbf{\Psi} = \begin{pmatrix} \psi_{11} & \\ \psi_{21} & \psi_{22} \end{pmatrix},$$

the covariance matrix of ϵ, a diagonal matrix,

$$\mathbf{\Theta}_\epsilon = \mathrm{diag}(\theta_{11}^{(\epsilon)}, \theta_{22}^{(\epsilon)}, \ldots, \theta_{44}^{(\epsilon)}) \ ,$$

and the covariance matrix of δ, also a diagonal matrix,

$$\mathbf{\Theta}_\delta = \mathrm{diag}(\theta_{11}^{(\delta)}, \theta_{22}^{(\delta)}, \ldots, \theta_{77}^{(\delta)}) \ .$$

1.3 LISREL submodels

Various submodels of LISREL are obtained by setting the numbers of certain variables to zero. (Numbers not specified are zero by default.)